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Soft sets and soft rings

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ABSTRACT

Molodtsov (1999) introduced the concept of soft sets in [1]. Then, Maji et al. (2003) defined some operations on soft sets in [2]. Aktaş and Çağman (2007) defined the notion of soft groups in [3]. Finally, soft semirings are defined by Feng et al. (2008) in [5]. In this paper, we have introduced initial concepts of soft rings.

1. Introduction

Dealing with uncertainties is a major problem in many areas such as economics, engineering, environmental science, medical science and social sciences. These kinds of problems cannot be dealt with by classical methods, because classical methods have inherent difficulties. To overcome these kinds of difficulties, Molodtsov [1] proposed a completely new approach, which is called soft set theory, for modeling uncertainty. Then Maji et al. [2] introduced several operations on soft sets. Aktaş and Çağman [3] defined soft groups and obtained the main properties of these groups. Moreover, they compared soft sets with fuzzy sets and rough sets. Besides, Jun et al. [4] defined soft ideals on BCK/BCI-algebras. Feng et al. [5] defined soft semirings, soft ideals on soft semirings and idealistic soft semirings. Qiu-Mei Sun et al. [6] defined the concept of soft modules and studied their basic properties.

The main purpose of this paper is to introduce basic notions of soft rings, which are actually a parametrized family of subrings of a ring, over a ring R. Moreover, the concept of the soft ring homomorphism is introduced and illustrated with a related example.

2. Preliminaries

In this section, we give some basic definitions for soft sets, mainly following [2].

Throughout this paper, U denotes an initial universe set and E is a set of parameters; the power set of U is denoted by $\mathcal{P}(U)$ and A is a subset of E.

Definition 2.1. A pair (F, A) is called a *soft set* over U, where F is a mapping given by $F : A \longrightarrow \mathcal{P}(U)$.

Definition 2.2. Let (F, A) and (G, B) be soft sets over a common universe U. Then (G, B) is called a *soft subset* of (F, A) if it satisfies the following:

(1) $B \subset A$.
(2) For all $x \in B$, $F(x)$ and $G(x)$ are identical approximations.

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Definition 2.3. Let (F, A) and (G, B) be two soft sets over a common universe U. The *intersection* of (F, A) and (G, B) is defined as the soft set (H, C) satisfying the following conditions:

(1) $C = A \cap B$.
(2) For all $x \in C$, $H(x) = F(x)$ or $G(x)$ (while the two sets are the same).

In this case, we write $(F, A)\widetilde{\cap}(G, B) = (H, C)$.

Definition 2.4. Let (F, A) and (G, B) be two soft sets over a common universe U. The *bi-intersection* of (F, A) and (G, B) is defined as the soft set (H, C) satisfying the following conditions:

(1) $C = A \cap B$.
(2) For all $x \in C$, $H(x) = F(x) \cap G(x)$.

This is denoted by $(F, A)\widetilde{\sqcap}(G, B) = (H, C)$.

Definition 2.5. Let (F, A) and (G, B) be two soft sets over a common universe U. The *union* of (F, A) and (G, B) is defined as the soft set (H, C) satisfying the following conditions:

(1) $C = A \cup B$.
(2) For all $x \in C$,

$$H(x) = \begin{cases} F(x) & \text{if } x \in A - B, \\ G(x) & \text{if } x \in B - A, \\ F(x) \cup G(x) & \text{if } x \in A \cap B. \end{cases}$$

This is denoted by $(F, A)\,\widetilde{\cup}(G, B) = (H, C)$.

Definition 2.6. Let $(F_i, A_i)_{i \in I}$ be a nonempty family of soft sets over a common universe U. The *union* of these soft sets is defined as the soft set (H, C) satisfying the following conditions:

(1) $C = \bigcup_{i \in I} A_i$.
(2) For all $x \in C$, $H(x) = \bigcup_{i \in I(x)} F_i(x)$ where $I(x) = \{i \in I \mid x \in A_i\}$.

This is denoted by $\widetilde{\bigcup}_{i \in I}(F_i, A_i) = (H, C)$.

Definition 2.7. If (F, A) and (G, B) are two soft sets over a common universe U, then "(F, A) *AND* (G, B)" denoted by $(F, A)\widetilde{\wedge}(G, B)$ is defined as $(F, A)\widetilde{\wedge}(G, B) = (H, C)$, where $C = A \times B$ and $H(x, y) = F(x) \cap G(y)$, for all $(x, y) \in C$.

Definition 2.8. If (F, A) and (G, B) are two soft sets over a common universe U, then "(F, A) *OR* (G, B)" denoted by $(F, A)\widetilde{\vee}(G, B)$ is defined as $(F, A)\widetilde{\vee}(G, B) = (H, C)$, where $C = A \times B$ and $H(x, y) = F(x) \cup G(y)$, for all $(x, y) \in C$.

For a soft set (F, A), the support of (F, A) is defined in [5]. We recall this definition.

Definition 2.9. Let (F, A) be a soft set. The set $\mathrm{Supp}(F, A) = \{x \in A \mid F(x) \neq \emptyset\}$ is called the *support* of the soft set (F, A). A soft set is said to be *non-null* if its support is not equal to the empty set.

3. Soft rings

From now on, R denotes a commutative ring and all soft sets are considered over R.

Definition 3.1. Let (F, A) be a non-null soft set over a ring R. Then (F, A) is called a *soft ring* over R if $F(x)$ is a subring of R for all $x \in A$.

Example 3.2. Let $R = A = \mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$. Consider the set-valued function $F : A \longrightarrow \mathcal{P}(R)$ given by $F(x) = \{y \in R \mid x.y = 0\}$. Then $F(0) = R$, $F(1) = \{0\}$, $F(2) = \{0, 3\}$, $F(3) = \{0, 2, 4\}$, $F(4) = \{0, 3\}$ and $F(5) = \{0\}$. As we see, all of these sets are subrings of R. Hence, (F, A) is a soft ring over R.

Theorem 3.3. *Let (F, A) and (G, B) be soft rings over R. Then:*

(1) *$(F, A)\widetilde{\wedge}(G, B)$ is a soft ring over R if it is non-null.*
(2) *The bi-intersection $(F, A)\widetilde{\sqcap}(G, B)$ is a soft ring over R if it is non-null.*

Proof. (1) By Definition 2.7, let $(F, A)\widetilde{\curlywedge}(G, B) = (H, C)$, where $C = A \times B$ and $H(a, b) = F(a) \cap G(b)$, for all $(a, b) \in C$. Since (H, C) is non-null, $H(a, b) = F(a) \cap G(b) \neq \emptyset$. Since the intersection of any number of subrings of R is a subring of R, $H(a, b)$ is a subring of R. Hence, (H, C) is a soft ring over R.
(2) By Definition 2.4, we have $(F, A)\widetilde{\sqcap}(G, B) = (H, C)$, where $H(x) = F(x) \cap G(x) \neq \emptyset$, for some $x \in A \cap B$. We observe that $F(x) \cap G(x)$ is a subring of R, since $H(x) \neq \emptyset$ and $F(x), G(x)$ are subrings of R. Consequently, $(H, C) = (F, A)\widetilde{\sqcap}(G, B)$ is a soft ring over R if it is non-null. $\square$

Definition 3.4. Let (F, A) and (G, B) be soft rings over R. Then (G, B) is called a *soft subring* of (F, A) if it satisfies the following:

(1) $B \subset A$.
(2) $G(x)$ is a subring of $F(x)$, for all $x \in \mathrm{Supp}(G, B)$.

Example 3.5. Let $R = A = 2\mathbb{Z}$ and $B = 6\mathbb{Z} \subset A$. Consider the set-valued functions $F : A \longrightarrow \mathcal{P}(R)$ and $G : B \longrightarrow \mathcal{P}(R)$ given by $F(x) = \{nx \mid n \in \mathbb{Z}\}$ and $G(x) = \{5nx \mid n \in \mathbb{Z}\}$. As we see, for all $x \in B$, $G(x) = 5x\mathbb{Z}$ is a subring of $x\mathbb{Z} = F(x)$. Hence, (G, B) is a soft subring of (F, A).

Theorem 3.6. *Let (F, A) and (G, B) be soft rings over R. Then we have the following:*

(1) *If $G(x) \subset F(x)$, for all $x \in B \subset A$, then (G, B) is a soft subring of (F, A).*
(2) *$(F, A)\widetilde{\sqcap}(G, B)$ is a soft subring of both (F, A) and (G, B) if it is non-null.*

Proof. (1) Clear.
(2) Let $(F, A)\widetilde{\sqcap}(G, B) = (H, C)$. Since $A \cap B \subset A$ and $H(x) = F(x) \cap G(x)$ is a subring of $F(x)$, (H, C) is a soft subring of (F, A). Similarly, we see that (H, C) is a soft subring of (G, B). $\square$

Example 3.7. Let $R = \mathbb{Z}$, $A = 2\mathbb{Z}$ and $B = 3\mathbb{Z}$. Consider the functions $F : A \longrightarrow \mathcal{P}(R)$ and $G : B \longrightarrow \mathcal{P}(R)$ defined by $F(x) = \{2nx \mid n \in \mathbb{Z}\} = 2x\mathbb{Z}$ and $G(x) = \{3nx \mid n \in \mathbb{Z}\} = 3x\mathbb{Z}$. Let $(F, A)\widetilde{\sqcap}(G, B) = (H, C)$ where $C = A \cap B = 6\mathbb{Z}$. For every $x \in C$, we have $H(x) = F(x) \cap G(x) = 6x\mathbb{Z}$ which is a subring of both $F(x) = 2x\mathbb{Z}$ and $G(x) = 3x\mathbb{Z}$. Consequently, $(F, A)\widetilde{\sqcap}(G, B)$ is a soft subring of both (F, A) and (G, B).

Theorem 3.8. *Let $(F_i, A_i)_{i \in I}$ be a nonempty family of soft rings over R. Then:*

(1) *$\widetilde{\bigwedge}_{i \in I}(F_i, A_i)$ is a soft ring over R if it is non-null.*
(2) *$\widetilde{\sqcap}_i(F_i, A_i)$ is a soft ring over R if it is non-null.*
(3) *If $\{A_i \mid i \in I\}$ are pairwise disjoint, then $\widetilde{\bigcup}_{i \in I}(F_i, A_i)$ is a soft ring over R.*

Proof. (1) Similar to the proof of Theorem 3.7 in [5].
(2) It is obvious since the intersection of any number of subrings of a ring is a ring.
(3) Result of Definition 2.5. $\square$

4. The soft ideal of a soft ring

In classical algebra, the notion of ideals is very important. For this reason, we introduce the soft ideals of a soft ring. Note that, if I is an ideal of a ring R, we write $I \lhd R$.

Definition 4.1. Let (F, A) be a soft ring over R. A non-null soft set (γ, I) over R is called *soft ideal* of (F, A), which will be denoted by $(\gamma, I)\widetilde{\lhd}(F, A)$, if it satisfies the following conditions:

(1) $I \subset A$.
(2) $\gamma(x)$ is an ideal of $F(x)$ for all $x \in \mathrm{Supp}(\gamma, I)$.

Example 4.2. Let $R = A = \mathbb{Z}_4 = \{0, 1, 2, 3\}$ and $I = \{0, 1, 2\}$.
Let us consider the set-valued function $F : A \longrightarrow \mathcal{P}(R)$ given by $F(x) = \{y \in R \mid x.y \in \{0, 2\}\}$. Then $F(0) = R$, $F(1) = \{0\}, F(2) = \mathbb{Z}_4$ and $F(3) = \{0, 2\}$. As we see, all these sets are subrings of R. Hence, (F, A) is a soft ring over R. On the other hand, consider the function $\gamma : I \longrightarrow \mathcal{P}(R)$ given by $\gamma(x) = \{y \in R \mid x.y = 0\}$. As we see, $\gamma(0) = R \lhd R$, $\gamma(1) = \{0\} \lhd F(1) = \{0\}$ and $\gamma(2) = \{0, 2\} \lhd F(2) = \mathbb{Z}_4$. Hence, (γ, I) is a soft ideal of (F, A).

Theorem 4.3. *Let (γ_1, I_1) and (γ_2, I_2) be soft ideals of a soft ring (F, A) over R. Then $(\gamma_1, I_1)\widetilde{\sqcap}(\gamma_2, I_2)$ is a soft ideal of (F, A) if it is non-null.*

Proof. Clear. $\square$

In Theorem 4.4, we have shown that the bi-intersection of two soft ideals of different soft rings is a soft ideal of the bi-intersection of these soft rings.

Theorem 4.4. *Let (γ_1, I_1) and (γ_2, I_2) be soft ideals of soft rings (F, A) and (G, B) over R respectively. Then $(\gamma_1, I_1)\widetilde{\sqcap}(\gamma_2, I_2)$ is a soft ideal of $(F, A)\widetilde{\sqcap}(G, B)$ if it is non-null.*

Proof. By Definition 2.4, we can write $(\gamma_1, I_1)\widetilde{\sqcap}(\gamma_2, I_2) = (\gamma, I)$, where $I = I_1 \cap I_2$ and $\gamma(x) = \gamma_1(x) \cap \gamma_2(x)$ for all $x \in I$. Similarly, we have $(F, A)\widetilde{\sqcap}(G, B) = (H, C)$ and $C = A \cap B$ where $H(x) = F(x) \cap G(x)$ for all $x \in C$. Since $I_1 \cap I_2$ is non-null, there exists an $x \in \mathrm{Supp}(\gamma, I)$ such that $\gamma(x) = \gamma_1(x) \cap \gamma_2(x) \neq \emptyset$. Since $I_1 \cap I_2 \subset A \cap B$, we need to show that $\gamma(x)$ is an ideal of ring $H(x)$ for all $x \in \mathrm{Supp}(\gamma, I)$. Because of the facts that $\gamma_1(x) \subset F(x)$ and $\gamma_2(x) \subset G(x)$, we see that $\gamma_1(x) \cap \gamma_2(x) \subseteq F(x) \cap G(x)$. Hence, $\gamma(x)$ is a subring of R. Finally we shall show that $r.a \in \gamma(x)$ for all $r \in H(x)$ and for all $a \in \gamma(x)$. Since $\gamma_1(x)$ is an ideal of $F(x)$, for $r \in H(x) = F(x) \cap G(x)$ and $a \in \gamma(x) = \gamma_1(x) \cap \gamma_2(x)$, we observe that $r.a \in \gamma_1(x)$ and $r.a \in \gamma_2(x)$. Hence, $r.a \in \gamma(x)$. □

Example 4.5. Let $R = M_2(\mathbb{Z})$, i.e., 2×2 matrices with integer terms, $A = 3\mathbb{Z}$, $B = 5\mathbb{Z}$, $I_1 = 6\mathbb{Z}$ and $I_2 = 10\mathbb{Z}$. Consider the functions $F : A \longrightarrow \mathscr{P}(R)$ and $G : B \longrightarrow \mathscr{P}(R)$ defined by

$$F(x) = \left\{\begin{bmatrix} nx & 0 \\ 0 & nx \end{bmatrix} \mid n \in \mathbb{Z}\right\} \quad \text{and} \quad G(x) = \left\{\begin{bmatrix} nx & nx \\ 0 & nx \end{bmatrix} \mid n \in \mathbb{Z}\right\}$$

which are subrings of R. Thus, (F, A) and (G, B) are soft rings over R. Consider the set-valued functions $\gamma_1 : I_1 \to P(R)$ and $\gamma_2 : I_2 \to P(R)$ defined by

$$\gamma_1(x) = \left\{\begin{bmatrix} nx & nx \\ 0 & 0 \end{bmatrix} \mid n \in \mathbb{Z}\right\} \quad \text{and} \quad \gamma_2(x) = \left\{\begin{bmatrix} 0 & nx \\ 0 & nx \end{bmatrix} \mid n \in \mathbb{Z}\right\}$$

which are ideals of $F(x)$ and $G(x)$ respectively. For all $x \in I_1 \cap I_2$,

$$\gamma_1(x) \cap \gamma_2(x) = \left\{\begin{bmatrix} 0 & nx \\ 0 & 0 \end{bmatrix} \mid n \in \mathbb{Z}\right\} \lhd F(x) \cap G(x) = \left\{\begin{bmatrix} nx & 0 \\ 0 & nx \end{bmatrix} \mid n \in \mathbb{Z}\right\}.$$

This indicates that $(\gamma_1, I_1)\widetilde{\sqcap}(\gamma_2, I_2)$ is a soft ideal of $(F, A)\widetilde{\sqcap}(G, B)$.

Theorem 4.6. *Let (F, A) be a soft ring over R and (γ_1, I_1), (γ_2, I_2) be soft ideals of (F, A) over R. If I_1 and I_2 are disjoint, then $(\gamma_1, I_1)\bigcup^{\sim}(\gamma_2, I_2)$ is a soft ideal of (F, A).*

Proof. According to Definition 2.5, $(\gamma_1, I_1)\bigcup^{\sim}(\gamma_2, I_2) = (\beta, I)$ where $I_1 \cup I_2 = I$ and for all $x \in I$,

$$\beta(x) = \begin{cases} \gamma_1(x) & \text{if } x \in I_1 - I_2, \\ \gamma_2(x) & \text{if } x \in I_2 - I_1, \\ \gamma_1(x) \cup \gamma_2(x) & \text{if } x \in I_1 \cap I_2. \end{cases}$$

Since $(\gamma_1, I_1)\widetilde{\lhd}(F, A)$ and $(\gamma_2, I_2)\widetilde{\lhd}(F, A)$ we see that $I \subset A$. For every $x \in \mathrm{Supp}(\beta, I)$, $x \in I_1 - I_2$ or $x \in I_2 - I_1$, since I_1 and I_2 are disjoint. If $x \in I_1 - I_2$, then $\beta(x) = I_1(x) \neq \emptyset$ is an ideal of $F(x)$ since $(\gamma_1, I_1)\widetilde{\lhd}(F, A)$. Similarly, if $x \in I_2 - I_1$, then $\beta(x) = I_2(x) \neq \emptyset$ is an ideal of $F(x)$ since $(\gamma_2, I_2)\widetilde{\lhd}(F, A)$. Thus, $\beta(x) \lhd F(x)$ for all $x \in \mathrm{Supp}(\beta, I)$. Hence, (β, I) is a soft ideal of (F, A). □

Theorem 4.7. *Let (F, A) be a soft ring over R and $(\gamma_k, I_k)_{k\in K}$ be a nonempty family of soft ideals of (F, A). Then we have the following:*

(1) *$\widetilde{\sqcap}_k(\gamma_k, I_k)$ is a soft ideal of (F, A) if it is non-null.*
(2) *$\widetilde{\bigwedge}_{k\in I}(\gamma_k, I_k)$ is a soft ideal of (F, A) if it is non-null.*
(3) *If $\{I_i \mid k \in K\}$ are pairwise disjoint, then $\widetilde{\bigcup}_{k\in I}(\gamma_k, I_k)$ is a soft ideal of (F, A) if it is non-null.*

Proof. (1) It is an obvious result since the intersection of an arbitrary nonempty family of ideals of a ring is an ideal of it. (2) and (3) are similar to (1). □

5. Idealistic soft rings

Definition 5.1. Let (F, A) be a non-null soft set over R. Then (F, A) is called an *idealistic soft ring* over R if $F(x)$ is an ideal of R for all $x \in \mathrm{Supp}(F, A)$.

Example 5.2. In Example 4.2, (F, A) is an idealistic soft ring over R since $F(x)$ is an ideal of R for all $x \in A$.

Proposition 5.3. *Let (F, A) be a soft set over R and $B \subset A$. If (F, A) is an idealistic soft ring over R, then so is (F, B) whenever it is non-null.*

Proof. Obvious. □

Theorem 5.4. *Let (F, A) and (G, B) be idealistic soft rings over R. Then $(F, A)\widetilde{\sqcap}(G, B)$ is an idealistic soft ring over R if it is non-null.*

Proof. By Definition 2.4, we have $(F, A)\widetilde{\sqcap}(G, B) = (H, C)$ where $C = A \cap B$ and $H(x) = F(x) \cap G(x)$, for all $x \in C$. Assume that (H, C) is a non-null soft set over R. So, if $x \in \mathrm{Supp}(H, C)$ then $H(x) = F(x) \cap G(x) \neq \emptyset$ and the nonempty sets $F(x)$ and $G(x)$ are ideals of R. Therefore, since the intersection of any nonempty family of ideals of a ring is an ideal of it, $H(x)$ is an ideal of R for all $x \in \mathrm{Supp}(H, C)$. Consequently, $(H, C) = (F, A)\widetilde{\sqcap}(G, B)$ is an idealistic soft ring over R. □

Theorem 5.5. *Let (F, A) and (G, B) be idealistic soft rings over R. If A and B are disjoint, then $(F, A)\,\widetilde{\cup}(G, B)$ is an idealistic soft ring over R.*

Proof. By Definition 2.5, $(F, A)\,\widetilde{\cup}(G, B) = (H, C)$ where $C = A \cup B$ and for all $x \in C$,

$$H(x) = \begin{cases} F(x) & \text{if } x \in A - B, \\ G(x) & \text{if } x \in B - A, \\ F(x) \cup G(x) & \text{if } x \in A \cap B. \end{cases}$$

Assume that $A \cap B = \emptyset$. Under this assumption, if $x \in \mathrm{Supp}(H, C)$ then $x \in A - B$ or $x \in B - A$. If $x \in A - B$, then $H(x) = F(x)$ is an ideal of R since (F, A) is an idealistic soft ring over R. Similarly, if $x \in B - A$, then $H(x) = G(x)$ is an ideal of R since (G, A) is an idealistic soft ring over R. Hence, for all $x \in \mathrm{Supp}(H, C)$, $H(x)$ is an ideal of R. As a result, $(H, C) = (F, A)\,\widetilde{\cup}(G, B)$ is an idealistic soft ring over R. □

In Theorem 5.5, if A and B are not disjoint, then the result is not true in general, because the union of two different ideals of a ring R may not be an ideal of R. See Example 5.6.

Example 5.6. Let $R = \mathbb{Z}_{10} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$, $A = \{0, 4\}$ and $B = \{4\}$. Consider the set-valued function $F : A \longrightarrow \mathscr{P}(R)$ given by $F(x) = \{y \in R \mid x.y = 0\}$. Then $F(0) = R \lhd R$ and $F(4) = \{0, 5\} \lhd R$. Hence, (F, A) is an idealistic soft ring over R. Now, consider the function $G : B \longrightarrow \mathscr{P}(R)$ given by $G(x) = \{0\} \cup \{y \in R \mid x + y \in \{0, 2, 4, 6, 8\}\}$. As we see, $G(4) = \{0, 2, 4, 6, 8\} \lhd R$. Therefore, (G, B) is an idealistic soft ring over R. Since $F(4) \cup G(4) = \{0, 2, 4, 5, 6, 8\}$ is not an ideal of R, $(F, A)\,\widetilde{\cup}(G, B)$ is not an idealistic soft ring over R.

Theorem 5.7. *Let (F, A) and (G, B) be idealistic soft rings over R. Then $(F, A)\widetilde{\wedge}(G, B)$ is an idealistic soft ring over R if it is non-null.*

Proof. By Definition 2.7, we have $(F, A)\widetilde{\wedge}(G, B) = (H, C)$, where $C = A \times B$ and $H(a, b) = F(a) \cap G(b)$, for all $(a, b) \in C$. Assume that (H, C) is a non-null soft set over R. If $(x, y) \in \mathrm{Supp}(H, C)$, then $H(x, y) = F(x) \cap G(y) \neq \emptyset$. Since (F, A) and (G, B) are idealistic soft rings over R, the nonempty sets $F(x)$ and $G(x)$ are ideals of R. Therefore, being an intersection of two ideals, $H(x, y)$ is an ideal of R for all $(x, y) \in \mathrm{Supp}(H, C)$. Consequently, $(H, C) = (F, A)\widetilde{\wedge}(G, B)$ is an idealistic soft ring over R. □

Example 5.8. Let $R = \left\{\begin{bmatrix} x & y \\ 0 & z \end{bmatrix} \mid x, y, z \in \mathbb{Z}\right\}$, $A = 6\mathbb{Z}$ and $B = 10\mathbb{Z}$. Consider the functions $F : A \longrightarrow \mathscr{P}(R)$ and $G : B \longrightarrow \mathscr{P}(R)$ defined by

$$F(x) = \left\{\begin{bmatrix} nx & nx \\ 0 & 0 \end{bmatrix} \mid n \in \mathbb{Z}\right\} \quad \text{and} \quad G(x) = \left\{\begin{bmatrix} 0 & nx \\ 0 & nx \end{bmatrix} \mid n \in \mathbb{Z}\right\}.$$

(F, A) and (G, B) are idealistic soft rings over R. Let $(F, A)\widetilde{\wedge}(G, B) = (H, C)$ where $C = A \times B$. Then, for all $(x, y) \in C$, we have

$$H(x, y) = F(x) \cap G(y) = \left\{\begin{bmatrix} 0 & tn \\ 0 & 0 \end{bmatrix} \mid n \in \mathbb{Z}\right\} \lhd R$$

where t is equal to the least common multiple of x and y.

Definition 5.9. An idealistic soft ring (F, A) over a ring R is said to be *trivial* if $F(x) = \{0\}$ for every $x \in A$. An idealistic soft ring (F, A) over R is said to be *whole* if $F(x) = R$ for all $x \in A$.

Example 5.10. Let p be a prime integer, $R = \mathbb{Z}_p$ and $A = \mathbb{Z}_p - \{0\}$. Consider the set-valued function $F : A \longrightarrow \mathscr{P}(R)$ given by $F(x) = \{y \in R \mid (x.y)^{p-1} = 1\} \cup \{0\}$. Then for all $x \in A$, we have $F(x) = R \lhd R$. Hence, (F, A) is a whole idealistic soft ring over R. Now, consider the function $G : A \longrightarrow \mathscr{P}(R)$ given by $G(x) = \{y \in R \mid xy = 0\}$. As we see, for all $x \in A$ we have $G(x) = \{0\} \lhd R$. Hence, (G, A) is a trivial idealistic soft ring over R.

Let (F, A) be a soft set over R and $f : R \longrightarrow R'$ be a mapping of rings. Then we can define a soft set $(f(F), A)$ over R' where $f(F) : A \longrightarrow \mathscr{P}(R')$ is defined as $f(F)(x) = f(F(x))$ for all $x \in A$. Here, by definition, we see that $\mathrm{Supp}(f(F), A) = \mathrm{Supp}(F, A)$.

Proposition 5.11. *Let $f : R \longrightarrow R'$ be a ring epimorphism. If (F, A) is an idealistic soft ring over R, then $(f(F), A)$ is an idealistic soft ring over R'.*

Proof. Since (F, A) is a non-null soft set by Definition 5.1 and (F, A) is an idealistic soft ring over R, we observe that $(f(F), A)$ is a non-null soft set over R'. We see that, for all $x \in \mathrm{Supp}(f(F), A)$, $f(F)(x) = f(F(x)) \neq \emptyset$. Since the nonempty set $F(x)$ is an ideal of R and f is an epimorphism, $f(F(x))$ is an ideal of R'. Therefore, $f(F(x))$ is an ideal of R' for all $x \in \mathrm{Supp}(f(F), A)$. Consequently, $(f(F), A)$ is an idealistic soft ring over R'. $\square$

Theorem 5.12. *Let (F, A) be an idealistic soft ring over R and $f : R \longrightarrow R'$ be a ring epimorphism.*

(1) *If $F(x) = \ker(f)$ for all $x \in A$, then $(f(F), A)$ is the trivial idealistic soft ring over R'.*
(2) *If (F, A) is whole, then $(f(F), A)$ is the whole idealistic soft ring over R'.*

Proof. (1) Suppose that $F(x) = \ker(f)$ for all $x \in A$. Then $f(F)(x) = f(F(x)) = \{0_{R'}\}$ for all $x \in A$. So, $(f(F), A)$ is the trivial idealistic soft ring over R' by Proposition 5.11 and Definition 5.9.

(2) Assume that (F, A) is whole. Then $F(x) = R$ for all $x \in A$. Hence, $f(F)(x) = f(F(x)) = f(R) = R'$ for all $x \in A$. As a result, by Proposition 5.11 and Definition 5.9, $(f(F), A)$ is the whole idealistic soft ring over R'. $\square$

Definition 5.13. Let (F, A) and (G, B) be soft rings over the rings R and R' respectively. Let $f : R \longrightarrow R'$ and $g : A \longrightarrow B$ be two mappings. The pair (f, g) is called a *soft ring homomorphism* if the following conditions are satisfied:

(1) f is a ring epimorphism,
(2) g is surjective,
(3) $f(F(x)) = G(g(x))\ \forall x \in A$.

If we have a soft ring homomorphism between (F, A) and (G, B), (F, A) is said to be soft homomorphic to (G, B), which is denoted by $(F, A) \sim (G, B)$. In addition, if f is a ring isomorphism and g is bijective mapping, then (f, g) is called a soft ring isomorphism. In this case, we say that (F, A) is softly isomorphic to (G, B), which is denoted by $(F, A) \simeq (G, B)$.

Example 5.14. Consider the rings $R = \mathbb{Z}$ and $R' = \{0\} \times \mathbb{Z}$. Let $A = 2\mathbb{Z}$ and $B = \{0\} \times 6\mathbb{Z}$. We see that (F, A) is a soft ring over R and (G, B) is a soft ring over R'. Consider the set-valued functions $F : A \longrightarrow \mathcal{P}(R)$ and $G : B \longrightarrow \mathcal{P}(R')$ which are given by $F(x) = x18\mathbb{Z}$ and $G((0, y)) = \{0\} \times 6y\mathbb{Z}$. Then the function $f : R \longrightarrow R'$ which is given by $f(x) = (0, x)$ is a ring isomorphism. Moreover, the function $g : A \longrightarrow B$ which is defined by $g(y) = (0, 3y)$ is a surjective map. As we see, for all $x \in A$, we have $f(F(x)) = f(18x\mathbb{Z}) = \{0\} \times 18x\mathbb{Z}$ and $G(g(x)) = G(\{0\} \times 6x\mathbb{Z}) = \{0\} \times 18x\mathbb{Z}$. Consequently, (f, g) is a soft ring isomorphism and $(F, A) \simeq (G, B)$.

6. Conclusion

The soft set concept and some basic algebraic structures on it are introduced by Molodtsov, Aktaş and Çağman, Maji et al., Jun et al., Feng et al., etc. In this paper, we defined soft rings and have introduced their initial basic properties such as soft ideals, soft homomorphisms etc. by using soft set theory. One may consider further algebraic structures of soft rings.

References

[1] D. Molodtsov, Soft set theory—first result, Comput. Math. Appl. 37 (1999) 19–31.
[2] P.K. Maji, R. Biswas, A.R. Roy, Soft set theory, Comput. Math. Appl. 45 (2003) 555–562.
[3] H. Aktas, N. Çağman, Soft sets and soft groups, Inform. Sci. 177 (2007) 2726–2735.
[4] Y.B. Jun, Soft BCK/BCI-algebra, Comput. Math. Appl. 56 (2008) 1408–1413.
[5] F. Feng, Y.B. Jun, X. Zhao, Soft semirings, Comput. Math. Appl. 56 (2008) 2621–2628.
[6] Qiu-Mei Sun, Zi-Liong Zhang, Jing Liu, Soft sets and soft modules, Lecture Notes in Comput. Sci. 5009 (2008) 403–409.